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Coupling between a surface-wave spectrum and an internal wave: modulational interaction

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Using a simple three-layer model of the ocean, we study a generation mechanism for the lowest internal-wave mode by nonlinear coupling to modulations of the surface-wave spectrum. We first examine the case of a narrow-band surface-wave spectrum, applying a method developed by Alber (1978) to derive a transport equation for the spectral density. Alber demonstrated that, when the spectral width (in the main wave direction) exceeds some critical value, the spectrum is stable against modulational perturbation (i.e. the Benjamin–Feir-type instability is suppressed). We show, however, that, for a stratified ocean, a modulational instability may persist because of a coupling between a 'modulational mode' of the surface-wave spectrum and an internal wave. The growth rate is calculated for a simple model of the angular distribution of the spectrum. It turns out that an important parameter is $\langle(\nabla\zeta)^2\rangle/\Delta\theta$, the ratio between the averaged square of the wave steepness, and the angular width of the spectrum.

For appreciable growth one must have roughly

$$2\times 10^{-3} \lesssim kd\langle(\nabla\zeta)^2\rangle/\Delta\theta,$$

where k is a characteristic wavenumber for the surface-wave spectrum, and d is the depth of the thermocline (50–100 m). This condition is probably too limiting for the above-mentioned modulational instability to be of any practical interest in the oceans.

We also consider the broad-band case of modulational interaction, and show the connection with incoherent three-wave interactions.

1. Introduction

In recent years a number of interesting papers have appeared on nonlinear interaction of surface waves with internal waves. The possibility that some such interaction may play a major part in maintaining the observed spectrum of internal waves in the oceans, still seems to be open (see Thorpe (1975) for a review on this topic).

The suggested mechanisms for nonlinear interaction seem to fall into three categories:

(*a*) coherent three-wave interactions;

(*b*) incoherent (or statistical) three-wave interactions;

(*c*) modulational interactions.

Theoretical models for (*a*) with two surface waves interacting with an internal wave, have been developed by Ball (1964), Thorpe (1966), Nesterov (1972), Brekhovskikh *et al.* (1972), and others. Tank experiments have been reported by Lewis, Lake & Ko (1974), and Joyce (1974). A more recent paper by Watson, West & Cohen (1976) has investigated (*a*) for a fairly realistic ocean model. They take into account a discretized representation of a surface-wave spectrum. They do, however, stick to the coherent interaction model in the sense that each surface-wave pair participating in a resonant trio (with an internal wave) has a perfectly deterministic phase relationship.

A theoretical model for (*b*) seems to have been first put forward by Hasselmann (1966, 1967) in a quite general form. A more detailed study of energy transfer from a swell spectrum to the lowest internal-wave mode in shallow water was reported by Kenyon (1968). Recently Olbers & Herterich (1979) have reported a rather comprehensive study of (*b*) for deep water, using a three-layer model of the ocean.

While the elementary processes behind (*a*) and (*b*) are more or less intuitively clear, (*c*) calls for an explanation.

The association of internal waves and the regular banded patterns of either surface slicks or surface roughness, often observed in coastal waters, is well documented (Ewing 1950; LaFond 1962; Perry & Schimke 1965). It was pointed out by Gargett & Hughes (1972) and Phillips (1973) that the presence of an internal wave would tend to modulate a surface-wave spectrum. This is simply due to the non-uniform surface current induced by the internal wave, which is refracting the surface waves. It has also been demonstrated in a laboratory experiment by Lewis *et al.* (1974). Although the theoretical papers mentioned above have used models where the internal wave (or rather the surface current induced by it) is considered to be given, the interaction is of course mutual in the sense that a non-uniform surface-wave spectrum influences the internal wave. To see this we note that non-uniformities in the surface-wave spectrum introduce corresponding non-uniformities in the radiation stress (see Longuet-Higgins & Stewart 1964), and thus surface forces acting on the internal wave (see figure 1). A similar type of interaction has been demonstrated between surface waves and Langmuir circulation by Garrett (1976). It follows from the description of (*c*) above that it only applies to the situation where the wavelength λ_I of the internal wave is much longer than a characteristic wavelength λ_S of the surface-wave spectrum.

The mechanism (*a*) is relatively strong, and predictions of the growth times of internal waves is of the order of a few hours (see Watson *et al.* 1976). This presupposes, however, that each two surface waves, mixing to produce an internal wave, have a deterministic phase relationship for as long as it takes to drive up the internal wave. This is equivalent to a requirement that the coherence time of the surface-wave spectrum should be of the order of the growth time (i.e. a few hours), which seems to be asking for a rather extreme situation.

The mechanism (*b*) does not rely on the assumption of a very long coherence time. Olbers & Herterich (1979) conclude that the interaction is of no importance for the deep and diffuse main thermocline of the ocean. For internal waves trapped in the seasonal (shallow) thermocline, however, they find a considerable coupling to the

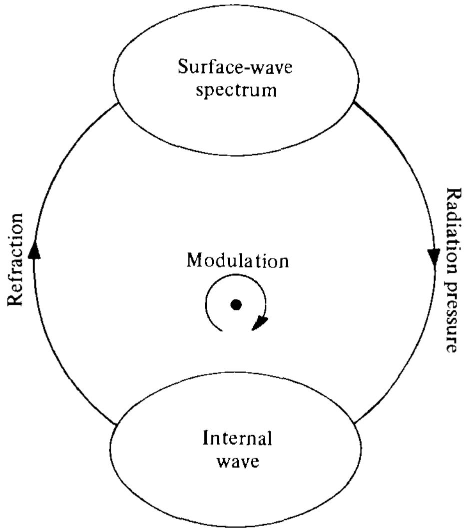

FIGURE 1

surface-wave spectrum. Even for moderate surface-wave conditions, they find a characteristic transfer time of the order of a day. In extreme situations such as strong storms or crossing swell components of high amplitude, the time scale will be a fraction of a day.

In this paper we investigate the mechanism (*c*). The most basic question in connexion with that mechanism is whether the 'feedback loop' it represents (as indicated in figure 1) is a positive one. In other words, will both the modulation and the internal wave grow as a result of the interaction? The findings to be reported here seem to indicate that the answer to the question is yes for quite general wave fields. We find, however, that the corresponding growth rate is very small indeed, except for the case when the angular width of the surface-wave spectrum is small enough to allow the existence of 'modulational modes' across the main direction of the wave field. The main effort of this paper is therefore concentrated on narrow-band spectra.

We remark that the mechanisms (*b*) and (*c*) are not, of course, entirely different. It is shown in §6 that, in the broad-band case, the modulational effect is included in a treatment based on (*b*) (e.g. Olbers & Herterich 1979) in the limit $\lambda_I \gg \lambda_S$. In the narrow-band case, however, the modulational interaction cannot be described in the framework of incoherent three-wave interactions, due to the existence of modulational modes.

Since the nonlinear coupling between surface waves and internal waves seems to be most significant for the lowest internal mode, we have concentrated on this mode, choosing a simple three-layer model of the ocean, with a shallow thermocline region of varying density separating homogeneous regions above and below. Our model thus corresponds to the seasonal (shallow) thermocline.

The plan of this paper is as follows. In § 2 we introduce the basic assumptions and equations. In § 3 we use a method developed by Alber (1978) to derive transport equations for the two-point correlation function and the spectral density of surface waves.

In § 4 we investigate the stability of a homogeneous distribution of the surface-wave spectrum to small-amplitude disturbances. These disturbances are shown to consist of two weakly coupled modes: a modulational mode and an internal-wave mode (the lowest internal mode). The modulational mode is shown to be identical to that studied by Alber (1978). As pointed out by him, this mode goes unstable when the width of the surface-wave spectrum is less than some critical value. For spectral widths larger than the critical, the modulational mode is generally damped by phase mixing. A simple example of spectra supporting undamped modulational modes is given.

In § 5 mode coupling is considered between a modulational mode and an internal wave. It is shown that, when a certain resonance condition is satisfied, an instability develops. For a simple model of the angular distribution of the surface-wave spectrum which allows explicit calculations, it is shown that the conditions for instability can always be satisfied. For the same model we calculate the growth rate, and give some numerical examples.

In § 6 we consider the case of a broad-band spectrum of surface waves. We choose a simple adiabatic (or WKB) approach, corresponding to the papers of Gargett & Hughes (1972) and Phillips (1973). We find that, although an instability seems to exist for quite general spectra, the growth rate is discouragingly small. We also indicate how this broad-band case relates to the incoherent three-wave interaction, and show how one can find simplified spectral transfer equations in the adiabatic limit (i.e. when a typical wavelength of the internal waves is much larger than a typical wavelength of the surface waves).

2. Basic equations and assumptions

In dealing with the mutual interaction between surface waves and internal waves in the ocean, we shall make a number of assumptions. Being interested mainly in the so-called lowest internal mode (see Phillips 1977, p. 211),† the following simple model of the ocean is chosen for convenience:

(i) a mixed layer of thickness d;

(ii) below the mixed layer is a layer, T, of thickness d_1, with non-vanishing density gradient (thermocline region);

(iii) below the thermocline region the ocean is assumed to be homogeneous.

If a typical wavelength of the internal wave considered is λ_I, and a typical wavelength in the spectrum of surface waves is λ_S, we shall also make the following assumptions:

(iv) $\lambda_I \gg d_1$, $\lambda_S < d$, $\lambda_I \gg \lambda_S$;

(v) we assume the surface waves to have a rather narrow spectrum.

If z is the vertical co-ordinate, with $z = 0$ at the equilibrium level of the free surface, and $\zeta(x, y, t)$ the surface elevation, we take as a starting-point of our calculations the equations

$$\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial z}+\frac{\partial}{\partial t}(\nabla\phi)^2+\nabla\phi\,.\,\nabla\frac{(\nabla\phi)^2}{2} = 0, \quad z = \zeta, \tag{2.1}$$

$$\frac{\partial\zeta}{\partial t}+\nabla\phi\,.\,\nabla\zeta = \frac{\partial\phi}{\partial z}, \quad z = \zeta, \tag{2.2}$$

† It appears from a number of papers (e.g. Joyce 1974; Watson *et al.* 1976; Olbers & Herterich 1979) that the strongest interaction occurs for the lowest internal mode.

$$z \in T, \quad \frac{\partial^2}{\partial t^2}(\nabla^2 w) + N^2 \nabla_h^2 w = 0, \tag{2.3}$$

$$z \notin T, \quad \nabla^2 \phi = 0. \tag{2.4}$$

Here g is the acceleration of gravity, ϕ is the velocity potential, w is the vertical velocity component, $\nabla_h \equiv \{\partial/\partial x, \partial/\partial y, 0\}$, $N(z)$ is the local Brunt–Väisälä frequency ($N^2 = -(g/\rho)\, d\rho/dz$), and T denotes the thermocline region $-d > z > -d-d_1$.

Equation (2.1) is obtained by taking a total time derivative of the Bernoulli equation, and assuming that the atmospheric pressure is constant (see Phillips 1977, p. 34). Equation (2.2) is the kinematic boundary condition, and (2.4) is the condition of incompressibility (except in the region T where the potential flow assumption breaks down). Equation (2.3) is obtained if the Boussinesq approximation is applied (see Phillips 1977, p. 207), and the internal wave is assumed to have a sufficiently small amplitude for nonlinear terms to be neglected. Thus (2.3) implies an additional assumption to the ones listed previously. We shall next be concerned with internal waves growing as a result of an interaction with the surface-wave spectrum. By neglecting nonlinear terms, to obtain (2.3), we have effectively neglected interactions between internal waves, such as parametric decay of the lowest mode into waves of higher mode number (Davis & Acrivos 1967; Martin, Simmons & Wunsch 1972).

To develop equations (2.1) and (2.2) further, we shall assume that the surface waves have a rather narrow spectrum centred at the wave vector $\mathbf{k}$, and frequency $\omega(=(gk)^{\frac{1}{2}})$. In § 6 we shall be able to relax this condition. Next we assume the wave-number bandwidth to be of the order ϵk, where ϵ is a small number taken to be the r.m.s. value of $|\nabla \zeta|$ (wave steepness). This suggests the feasibility of the following expansions for ζ and ϕ

$$\begin{aligned} \zeta &= \bar{\zeta} + \tfrac{1}{2}[A\, e^{i\theta} + A_2 e^{2i\theta} + \ldots + \text{c.c.}], \\ \phi &= \bar{\phi} + \tfrac{1}{2}[B\, e^{kz+i\theta} + B_2 e^{2kz+2i\theta} + \ldots + \text{c.c.}], \end{aligned} \tag{2.5}$$

where $\theta = \mathbf{k}.\mathbf{x} - \omega t$, $k = |\mathbf{k}|$, and c.c. means complex conjugate. Here $\bar{\phi}$ and $\bar{\zeta}$ are real functions slowly varying in space and time, representing the perturbations brought about by (i) the internal wave, and (ii) by spacial non-uniformity of the surface-wave radiation stress (Longuet-Higgins & Stewart 1964). The complex coefficients $A, A_2, A_3, \ldots, B, B_2, B_3, \ldots$, are slowly varying on a time scale ϵt and a space scale $\epsilon \mathbf{x}$.†

In the absence of inhomogeneity (and thus internal waves) it is found (Dysthe 1979) that $\bar{\phi}$ is of the order ϵ^2. In the present ocean model ((i)–(iii) above), due to the presence of internal waves, we shall allow for $\bar{\phi} = O(\epsilon)$.‡ Developing (2.1) and (2.2) to the third order in ϵ, inserting (2.5) and using (2.4), one obtains after some manipulation the following equations at the surface

$$i\left(\frac{\partial A}{\partial t} + v_g \frac{\partial A}{\partial x}\right) + \frac{1}{2}\frac{dv_g}{dk}\frac{\partial^2 A}{\partial x^2} + \frac{v_g}{2k}\frac{\partial^2 A}{\partial y^2} = \tfrac{1}{2}k^2\omega |A|^2 A + kA\frac{\partial \bar{\phi}}{\partial x}, \tag{2.6}$$

$$\frac{\partial \bar{\phi}}{\partial z} + \frac{1}{g}\frac{\partial^2 \bar{\phi}}{\partial t^2} = \tfrac{1}{2}\omega \frac{\partial}{\partial x}|A|^2, \tag{2.7}$$

† In a co-ordinate system moving with the group velocity, the variation is on the time scale $\epsilon^2 t$.

‡ This ordering makes the lowest-order interaction term $kA\,\partial\bar{\phi}/\partial x$ in (2.6) below, of comparable order of magnitude to the 'self-interaction' term preceding it.

where we have arranged for the x-axis to be along $\mathbf{k}$. Here $v_g = g/2\omega$ is the group velocity and in (2.6) we have used the fact that $|\partial\overline{\phi}/\partial z| \ll |\partial\overline{\phi}/\partial x|$ at the surface $z = 0$.† The equation (2.7) is obtained by averaging the expanded version of (2.1) on the fast time scale. In both (2.6) and (2.7) terms of order ϵ^4 and higher have been neglected. Noting that (2.4) is also valid for $\overline{\phi}$, the equations (2.3), (2.4), (2.6) and (2.7) serve as our starting-point in the investigation of the interaction between surface waves and internal waves. In the absence of inhomogeneity (T vanishes and equation (2.3) is absent) the system (2.4), (2.6) and (2.7) was shown to give an improvement on the nonlinear Schrödinger equation, as far as prediction on stability is concerned (Dysthe 1979). In that case, as pointed out above, the mean-flow velocity potential $\overline{\phi}$ is only $O(\epsilon^2)$.

3. The transport equations

When the amplitudes A and B are narrow-band random functions, (2.6) is still valid. We proceed to find a transport equation for the spectral function corresponding to A by applying the method of Alber (1978).

Alber demonstrated that, if A was narrow-band and Gaussian distributed, the transport equations could be easily derived from an evolution equation for the amplitude such as (2.6). Using his method, we find that the two-point correlation function

$$\rho = \langle A(\mathbf{x}_1, t)\, A^*(\mathbf{x}_2, t)\rangle,$$

where $\langle\ \rangle$ means ensemble average, and $*$ denotes complex conjugate, satisfies the following transport equation

$$i\left(\frac{\partial\rho}{\partial t} + v_g\frac{\partial\rho}{\partial x}\right) + \frac{dv_g}{dk}\frac{\partial^2\rho}{\partial x\,\partial r_x} + \frac{v_g}{k}\frac{\partial^2\rho}{\partial y\,\partial r_y}$$

$$= k\rho\frac{\partial}{\partial x}[\overline{\phi}(\mathbf{x}+\tfrac{1}{2}\mathbf{r}) - \overline{\phi}(\mathbf{x}-\tfrac{1}{2}\mathbf{r})] + \omega k^2\rho[\overline{|A(\mathbf{x}+\tfrac{1}{2}\mathbf{r})|^2} - \overline{|A(\mathbf{x}-\tfrac{1}{2}\mathbf{r})|^2}], \qquad (3.1)$$

where

$$\mathbf{x} \equiv \{x, y, 0\} = \tfrac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2),$$

$$\mathbf{r} \equiv \{r_x, r_y, 0\} = \mathbf{x}_1 - \mathbf{x}_2$$

and

$$\overline{|A(\mathbf{x})|^2} = \rho(\mathbf{x}+\tfrac{1}{2}\mathbf{r}, \mathbf{x}-\tfrac{1}{2}\mathbf{r})|_{\mathbf{r}=0}.$$

Introducing the power-spectral density

$$F(\mathbf{p}, \mathbf{x}, t) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \rho(\mathbf{x}+\tfrac{1}{2}\mathbf{r}, \mathbf{x}-\tfrac{1}{2}\mathbf{r})\, e^{-i\mathbf{p}\cdot\mathbf{r}}\, d\mathbf{r}, \qquad (3.2)$$

where $\mathbf{p} \equiv (p_x, p_y, 0\}$, the transport equation for $F(\mathbf{p}, \mathbf{x}, t)$ is found by taking the Fourier transform with respect to $\mathbf{r}$ of equation (3.1), giving

$$\frac{\partial F}{\partial t} + \left(v_g + p_x\frac{dv_g}{dk}\right)\frac{\partial F}{\partial x} + p_y\frac{v_g}{k}\frac{\partial F}{\partial y} = 2\sin\left(\frac{1}{2}\frac{\partial}{\partial\mathbf{p}}\cdot\frac{\partial}{\partial\mathbf{x}}\right)F\left[\omega k^2\overline{|A|^2} + k\frac{\partial\overline{\phi}}{\partial x}\right], \qquad (3.3)$$

where the spacial derivatives of the sine operator are operating only on the terms in the square brackets (see Alber 1978).

† This follows from (2.7) by observing that the second term on the left-hand side, as well as the right-hand side, is $O(\epsilon^3)$, thus $\partial\overline{\phi}/\partial z = O(\epsilon^3)$ at $z = 0$, while $\partial\overline{\phi}/\partial x = O(\epsilon^2)$.

Equation (2.7) now becomes

$$\frac{\partial\bar{\phi}}{\partial z}+\frac{1}{g}\frac{\partial^2\bar{\phi}}{\partial t^2}=\frac{\omega}{2}\frac{\partial}{\partial x}\overline{|A|^2}=\frac{\omega}{2}\frac{\partial}{\partial x}\iint F(\mathbf{p},\mathbf{x},t)\,d\mathbf{p}. \tag{3.4}$$

The equations (3.3), (3.4), (2.3) and (2.4) describe the interaction between the narrow-band spectrum F and the internal waves. They are valid when the correlation length scale of the surface waves, L_{corr}, is larger than, or of the same order of magnitude as, a typical wavelength, λ_I, of the internal waves (or surface modulation), i.e.

$$L_{\text{corr}} \gtrsim \lambda_I. \tag{3.5}$$

From (3.4) it is clear that the interaction of the kind we have taken to describe, depends on a spacial non-uniformity of the spectrum $F(\mathbf{p},\mathbf{x},t)$, having a characteristic wavelength comparable to that of an internal wave. Why such non-uniformities should exist, and persist for a time long enough for an internal wave to grow, is hard to understand, unless it is created by some instability. This will be investigated in §§ 4–6, which deal with the stability properties of an initially uniform spectrum.

4. Stability analysis

4.1. *Dispersion relation*

It is readily seen from (2.3), (2.4), (3.3) and (3.4) that a basic solution is

$$F=F_0(\mathbf{p}),\quad \bar{\phi}=w=0. \tag{4.1}$$

This solution represents a homogeneous distribution of the surface-wave spectrum. The radiation-stress-induced force, represented by the right-hand side of (3.4), then vanishes, and thereby the coupling to the internal waves.†

To investigate the stability of the basic solution (4.1), we assume that $\bar{\phi}$ and w are small perturbations, and perturb F and $\overline{|A|^2}$ in the following manner

$$F(\mathbf{p},\mathbf{x},t)=F_0(p)+f_1(\mathbf{p},\mathbf{x},t),$$

$$\overline{|A|^2}=\overline{|A|_0^2}+\overline{|A|_1^2}=\iint F_0\,d\mathbf{p}+\iint f_1\,d\mathbf{p},$$

where all perturbed quantities are assumed small, such that (3.3) can be linearized.

Taking a Fourier component

$$\exp i(\boldsymbol{\kappa}\,.\,\mathbf{x}-\Omega t)$$

of the perturbed quantities, one obtains from (2.4) and (2.3), and the boundary condition $\partial\bar{\phi}/\partial z=0$ when $z\to-\infty$

$$\bar{\phi}=\begin{cases}A\cosh\kappa z-B\sinh\kappa z, & \text{above } T,\\ C\exp\kappa z, & \text{below } T,\end{cases} \tag{4.2}$$

and

$$\frac{\partial^2 w}{\partial z^2}+\kappa^2\left(\frac{N^2}{\Omega^2}-1\right)w=0 \tag{4.3}$$

where the factor $\exp i(\boldsymbol{\kappa}\,.\,\mathbf{x}-\Omega t)$ has been suppressed.

† There will still be a coupling to a higher order than that considered here, as shown, e.g., by Olbers & Herterich (1979).

To lowest order in the small quantity κd_1, we obtain from (4.3) the following jump conditions for w across the thermocline region (see Phillips 1977, p. 212)

$$w(-d)-w(-d-d_1) = 0,$$

$$\frac{\partial w}{\partial z}(-d)-\frac{\partial w}{\partial z}(-d-d_1) = -\frac{g\kappa^2}{\Omega^2}\frac{\delta\rho}{\rho}\,w(-d), \tag{4.4}$$

where

$$\frac{\delta\rho}{\rho} = -\frac{1}{g}\int_T N^2\,dz$$

is the relative density increase through the thermocline, typically of order 10^{-3} for the ocean. Inserting (4.2) in (4.4), A and C can be expressed in terms of B (again to lowest order in κd_1) as

$$A = -\frac{B}{H}\left[1+\left(1-\frac{g\kappa}{\Omega^2}\frac{\delta\rho}{\rho}\right)\coth\kappa d\right],$$

and

$$C = -\frac{B}{H}\frac{e^{\kappa d}}{\sinh\kappa d},$$

where

$$H(\Omega,\kappa) = 1+\coth\kappa d-\frac{g\kappa}{\Omega^2}\frac{\delta\rho}{\rho}. \tag{4.5}$$

As noted previously the second term on the left-hand side of (3.4) is formally of order ϵ^3 and should be neglected (for the situation considered in this paper the ratio between the first and second term is of the order $\delta\rho/\rho$). The boundary condition for $\overline{\phi}$ at $z = 0$ thus becomes

$$\frac{1}{\kappa}\frac{\partial\overline{\phi}}{\partial z}(0) = i\frac{\omega}{2}\frac{l}{\kappa}\,\overline{|A|_1^2} = -B. \tag{4.6}$$

We note that, in the absence of the surface-wave spectrum, the dispersion relation for a freely propagating internal wave of the lowest mode is $H(\Omega,\kappa) = 0$.

Using the results (4.2), (4.5) and (4.6) in the linearized version of (3.3), one finally obtains the following dispersion relation for Ω and $\boldsymbol{\kappa} \equiv \{l, m, 0\}$

$$H(\Omega,\kappa)\,[1+Q(\Omega,\boldsymbol{\kappa})] = \frac{l^2}{2k\kappa}Q(\Omega,\boldsymbol{\kappa})\left[1+\left(1-\frac{g\kappa}{\Omega^2}\frac{\delta\rho}{\rho}\right)\coth\kappa d\right], \tag{4.7}$$

where

$$Q(\Omega,\boldsymbol{\kappa}) = \omega k^2\int\!\!\int_{-\infty}^{\infty}\frac{F_0(\mathbf{p}+\boldsymbol{\kappa}/2)-F_0(\mathbf{p}-\frac{1}{2}\boldsymbol{\kappa})}{\Omega-v_g l-p_x l\,dv_g/dk-p_y m v_g/k}\,d\mathbf{p}$$

and $H(\Omega,\kappa)$ is defined in (4.5). The integral is defined for $\operatorname{Im}\Omega > 0$ (see Alber 1978), i.e. for wave instability. When $\operatorname{Im}\Omega < 0$, $Q(\Omega,\boldsymbol{\kappa})$ must be interpreted as the analytic continuation of the function into the lower half of the complex Ω plane (cf. appendix).

Since the denominator of the integral Q only contains a certain linear combination of p_x and p_y, Q can be reduced to a single integral by introduction of the variables p_ξ and p_η given by $p_\xi = \mathbf{p}\,.\mathbf{i}_\xi$, and $p_\eta = \mathbf{p}\,.\mathbf{i}_\eta$, where the unit vectors $\mathbf{i}_\xi$ and $\mathbf{i}_\eta$ are given by

$$\mathbf{i}_\xi = \frac{\{\cos\theta,\ -2\sin\theta,0\}}{(\cos^2\theta+4\sin^2\theta)^{\frac{1}{2}}},\quad \mathbf{i}_\eta = \frac{\{2\sin\theta,\cos\theta,0\}}{(\cos^2\theta+4\sin^2\theta)^{\frac{1}{2}}}, \tag{4.8}$$

and θ is the angle between $\mathbf{k}$ and $\boldsymbol{\kappa}$. Note that

$$\{l\, dv_g/dk,\ mv_g/k,\ 0\} = dv_g/dk\{l,\ -2m,\ 0\}$$

which is a vector parallel to $\mathbf{i}_\xi$, and $\mathbf{i}_\xi . \mathbf{i}_\eta = 0$. Q can now be rewritten as follows:

$$Q(\Omega, \kappa) = \omega k^2 \int_{-\infty}^{\infty} \frac{\tilde{F}_0(p_\xi + \frac{1}{2}\kappa_\xi) - \tilde{F}_0(p_\xi - \frac{1}{2}\kappa_\xi)}{\Omega - v_g l + \beta p_\xi} dp_\xi, \tag{4.9}$$

where

$$\kappa_\xi = \boldsymbol{\kappa} . \mathbf{i}_\xi, \quad \beta = \kappa \left|\frac{dv_g}{dk}\right| (\cos^2\theta + 4\sin^2\theta)^{\frac{1}{2}}$$

and

$$\tilde{F}_0(p_\xi) = \int_{-\infty}^{\infty} F_0(\mathbf{p})\, dp_\eta.$$

4.2. *Stability of surface-wave modulations*

The equation (4.7) has the form of a dispersion relation of two weakly coupled wave modes, where the coupling term on the right-hand side is $O(\epsilon)$. Thus to lowest order they decouple into a dispersion relation

$$H(\Omega, \kappa) = 0, \tag{4.10}$$

which describes the propagation of an internal wave of the lowest mode, and a dispersion relation

$$1 + Q(\Omega, \boldsymbol{\kappa}) = 0, \tag{4.11}$$

which we would expect to describe the surface-wave modulation. Comparing with Alber (1978), we find that this is indeed so.

Let us first, for the sake of completeness, recall some of the information which can be extracted from (4.11) (see Alber 1978).

In the limit when the spectral width tends to zero it is readily shown by inserting $F_0(\mathbf{p}) = \overline{|A|_0^2}\, \delta(\mathbf{p})$ in (4.11) that one obtains the well-known result for modulation instability of a monochromatic wave train of amplitude a_0:

$$\Omega = v_g l \pm (L(L + \omega(ka_0)^2))^{\frac{1}{2}} \tag{4.12}$$

if one makes the identification $2\overline{|A|_0^2} = a_0^2$, where L is defined as

$$L = \frac{\omega}{8}\frac{2m^2 - l^2}{k^2}. \tag{4.13}$$

According to (4.12) instability occurs ($\operatorname{Im}\Omega > 0$) when $\boldsymbol{\kappa} \equiv \{l, m, 0\}$ belongs to the domain between the hyperbola $2m^2 - l^2 = 8k^2(ka_0)^2$, and the asymptotes $l^2 = 2m^2$, on the $\boldsymbol{\kappa}$ plane. Note that one has stability with respect to all perturbation wavenumbers $\boldsymbol{\kappa}$ whose direction is such that the angle θ between $\mathbf{k}$ and $\boldsymbol{\kappa}$ is larger than 35·26°.

For the two-dimensional Gaussian spectrum

$$F_0(\mathbf{p}) = \frac{\overline{|A|_0^2}}{2\pi\sigma_\xi\sigma_\eta} \exp - \left[\frac{p_\xi^2}{2\sigma_\xi^2} + \frac{p_\eta^2}{2\sigma_\eta^2}\right]$$

Alber found that, as the spectral width σ_ξ increased, some critical value was finally reached where the modulational instability disappeared altogether.

It is somewhat inconvenient to refer the distribution $F_0(\mathbf{p})$ to the co-ordinate system ξ, η. Since the centre of our wave spectrum is at the wavenumber $\mathbf{k}$, and the x axis therefore is a preferred direction (henceforth referred to as the 'wave direction'), we shall restate some of Alber's finding for a double-Gaussian

$$F_0(p) = \frac{\overline{|A|_0^2}}{2\pi\sigma_x\sigma_y}\exp - \left[\frac{p_x^2}{2\sigma_x^2}+\frac{p_y^2}{2\sigma_y^2}\right]. \tag{4.14}$$

From (4.14) one easily finds that

$$\tilde{F}_0(p_\xi) = \frac{\overline{|A|_0^2}}{2\pi\sigma_\xi}\exp\left(-p_\xi^2/2\sigma_\xi^2\right),$$

where $\sigma_\xi^2 = (\sigma_x^2 + 4\sigma_y^2 tg^2\theta)/(1+4tg^2\theta)$, and θ is the angle between $\mathbf{k}$ and $\boldsymbol{\kappa}$.

One of Alber's conditions† for stability then becomes

$$\left(\frac{\sigma_x^2+4\sigma_y^2 tg^2\theta}{1-2tg^2\theta}\right)^{\frac{1}{2}} > 2k(k^2\overline{|A|_0^2})^{\frac{1}{2}}. \tag{4.15}$$

The condition (4.15) is satisfied for all angles θ ($< 35{\cdot}26°$) if and only if

$$\sigma_x > 2k(k^2\overline{|A|_0^2})^{\frac{1}{2}}. \tag{4.16}$$

Thus (4.16) is the condition for complete suppression of the modulational instability. The right-hand side of (4.16) is equal to $2^{-\frac{1}{2}}$ times the wavenumber l_{mod} of the fastest-growing mode along the 'wave direction', in the limit of vanishing spectral width. Thus (4.16) can be written

$$\sigma_x > l_{\text{mod}}/2^{\frac{1}{2}} \equiv \sigma_0. \tag{4.16a}$$

Another feature which comes out of Alber's analysis for the case of finite spectral width is that, if (4.16) is satisfied, the corresponding modulational perturbation is generally damped. This contrasts the case of a spectral line ($F_0(\mathbf{p}) = \overline{|A|_0^2}\,\delta(\mathbf{p})$), where a perturbation is either unstable or neutrally stable. The damping is due to phase mixing, and the mechanism is analogous to that of Landau damping of electrostatic Langmuir waves in a plasma of finite temperature.

Note from (4.16) that only σ_x, i.e. the spectral width along the 'wave direction', enters the stability criterion. This implies, for example, that σ_y can be arbitrarily small without upsetting the stability.

Lastly, under the assumption that the spectral width is large enough for the surface waves to be modulationally stable, we shall try to identify modulational perturbations that are likely candidates for interaction with the internal wave. To lowest order the frequency of the modulational perturbation of wave vector $\boldsymbol{\kappa}$ is $\mathbf{v}_g.\boldsymbol{\kappa}$.‡ Since the phase velocity of the internal wave is much smaller than v_g (for surface waves of reasonably long wavelength), interaction can only take place when θ, the angle between $\mathbf{k}$ and $\boldsymbol{\kappa}$, is near $\frac{1}{2}\pi$.

We also note that the coupling between the two modes is a very weak one, explicitly

† See Alber (1978), equation (5.3).

‡ This means that mode coupling occurs roughly when the component of the group velocity in the direction of modulation is equal to the phase speed of the internal wave.

demonstrated by the smallness of the coupling term on the right-hand side of (4.7).† For any significant interaction to be possible the modulational mode must have a negligible damping when considered as a 'free' mode (i.e. satisfying the dispersion relation (4.11)). It is not evident that such a modulational perturbation exists, so we try to demonstrate this by choosing an example which is simple enough for explicit calculations. Choosing for the spectral distribution the 'water bag' model

$$\tilde{F}_0(p_\xi) = \overline{|A|_0^2}\begin{cases} 1/2\sigma, & |p_\xi| \leqslant \sigma, \\ 0, & |p_\xi| > \sigma, \end{cases} \tag{4.17}$$

it is shown in the appendix that the dispersion relation (4.11) has an undamped solution, for $\kappa_\xi < 0$ (i.e. for $\theta > 35{\cdot}26°$) given by (A 4). It seems reasonable that one should get an undamped (or negligibly damped) modulational mode, also for a wider class of spectral distributions, having the characteristics that $\tilde{F}_0(p_\xi)$ is negligible outside some finite domain.

At this point it is perhaps appropriate to note that, since we are interested in modulations moving nearly perpendicular to the 'wave direction', $\tilde{F}_0(p_\xi)$ is actually the directional distribution of the wave spectrum.‡

For the directional distribution of swell at a long distance from the storm area, the above-mentioned models are perhaps not so unreasonable. This is because propagation over a large distance acts as a directional filter on the original distribution in the storm area (see, for example, Kinsman 1965, p. 405), as indicated by figure 2.

5. Mode coupling

Due to the smallness of the coupling term on the right-hand side of (4.7), the solutions of the dispersion relation are well approximated by the solutions of the two equations (4.10) and (4.11). Denoting a solution of the former by $\Omega_I(\kappa)$, and of the latter by $\Omega_M(\boldsymbol{\kappa})$, we note that the approximation breaks down near the points $\boldsymbol{\kappa}_0$ in the $\boldsymbol{\kappa}$ plane, where $\Omega_M(\boldsymbol{\kappa}_0) = \Omega_I(\kappa_0) \equiv \Omega_0$. This is exactly where the interaction, or mode coupling, takes place.

To obtain an approximate solution of (4.7) valid near Ω_0, $\boldsymbol{\kappa}_0$, we evaluate the right-hand side at Ω_0, $\boldsymbol{\kappa}_0$. On the left-hand side we keep only the first two terms of the Taylor expansions of $H(\Omega, \kappa)$ and $Q(\Omega, \boldsymbol{\kappa})$ around Ω_0, $\boldsymbol{\kappa}_0$. This gives the equation

$$(\bar{\omega} - v_{gI}\bar{k})\,(\bar{\omega} - \mathbf{v}_{gM}\cdot\bar{\mathbf{k}}) = \frac{l_0^2}{2k\kappa_0 \sinh^2(\kappa_0 d)\,\partial Q/\partial\Omega|_0\,\partial H/\partial Q|_0}, \tag{5.1}$$

where

$$v_{gI} = -\left(\frac{\partial H}{\partial \kappa}\bigg/\frac{\partial H}{\partial \Omega}\right)\bigg|_0, \quad V_{gM} = -\left(\frac{\partial Q}{\partial \boldsymbol{\kappa}}\bigg/\frac{\partial Q}{\partial \Omega}\right)\bigg|_0$$

† Note that the right-hand side of (4.7) is of order $\epsilon\cos^2\theta$, which is especially small when θ is near $\frac{1}{2}\pi$.

‡ Note that the distribution of the surface-wave spectrum along the 'wave direction' is of minor importance as long as it is broad enough to be stable. From (4.8) it is seen that, for $\theta = \frac{1}{2}\pi$, $\mathbf{i}_\xi$ is along the negative-y axis. Thus $\tilde{F}_0(p_\xi)$ gives the spectral distribution perpendicular to the 'wave direction'. For a narrow spectrum this is approximately the same as the directional distribution.

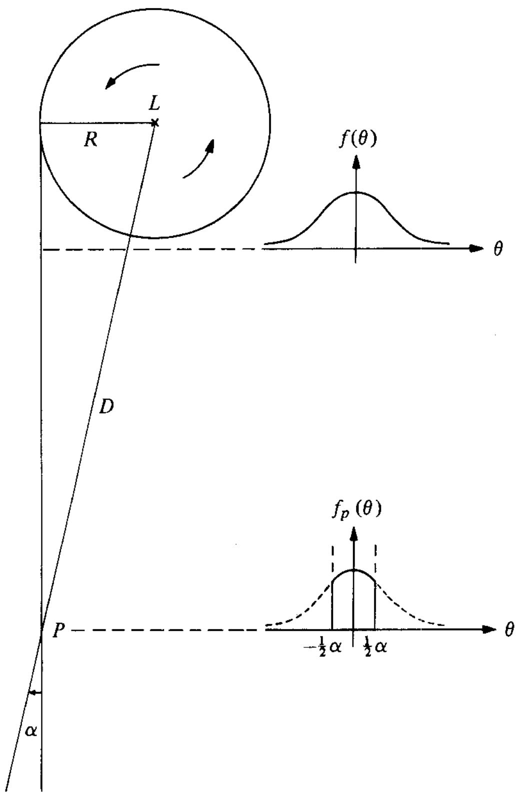

FIGURE 2. Sketch indicating how the directional spectrum near a storm area, $f(\theta)$, is 'filtered', by propagating a distance D, into a spectrum $f_p(\theta)$.

and

$$\bar{\omega} = \Omega - \Omega_0, \quad \bar{\mathbf{k}} = \boldsymbol{\kappa} - \boldsymbol{\kappa}_0.$$

Since

$$\partial H/\partial\Omega|_0 = (2g\kappa_0/\Omega_0^3)\,\delta\rho/\rho > 0$$

(5.1) predicts instability when

$$\partial Q/\partial\Omega|_0 < 0. \tag{5.2}$$

When (5.2) is satisfied, the maximum growth rate occurs for $\bar{\mathbf{k}} = 0$ (i.e. $\boldsymbol{\kappa} = \boldsymbol{\kappa}_0$), and is given by

$$\gamma = \left\{\frac{l_0^2\,\Omega_0}{4k\kappa_0\sinh^2(\kappa_0 d)\,(1+\coth\kappa_0 d)\,|\partial Q/\partial\Omega|_0|}\right\}^{\frac{1}{2}}. \tag{5.3}$$

For the simple 'waterbag' model (4.17) of the directional spectral distribution, it is readily shown that $\partial Q/\partial\Omega|_0$ is negative for the branch $\Omega_M^+(\boldsymbol{\kappa})$ (see A4)), and (5.3) can be calculated explicitly. Doing this, and taking into account that $|l_0/m_0| \ll 1$ as explained below, one obtains

$$\frac{\gamma}{\omega} = \frac{1}{4}\left(\frac{\delta\rho}{\rho}\right)^{\frac{1}{2}}(kd)^{-\frac{3}{4}}\left|\frac{l_0}{k}\right| G\left(c, \frac{\sigma}{m_0}\right) P(m_0 d), \tag{5.4}$$

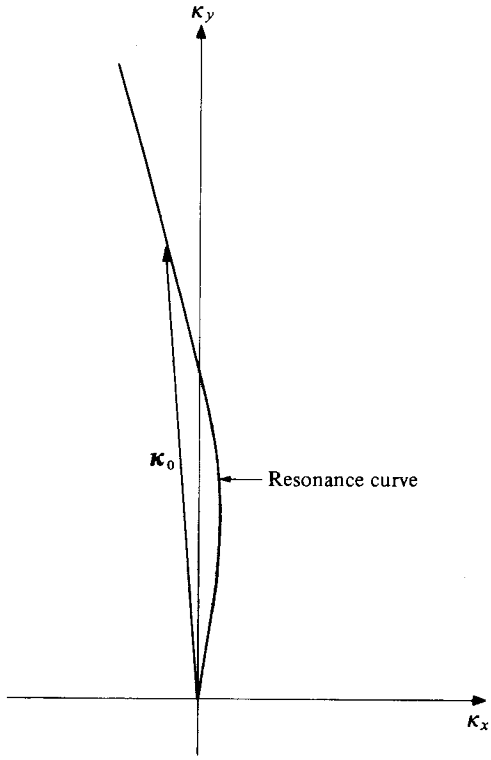

FIGURE 3. Sketch of resonance curve in the κ_x, κ_y plane for the modulational interaction.

where

$$G\left(c, \frac{\sigma}{m_0}\right) = \frac{(c\sigma/m_0)^{\frac{1}{2}}}{(\frac{1}{4}+(\sigma/m_0)^2+\sigma/m_0 \coth c)^{\frac{1}{2}} \sinh c},$$

$$P(m_0 d) = \frac{(m_0 d\, e^{-m_0 d})^{\frac{1}{2}}}{\sinh^{\frac{1}{2}} m_0 d},$$

and $c = 2\sigma m_0/\sigma_0^2$, where $\sigma_0 = 2k(k^2|A|_0^2)^{\frac{1}{2}}$ is the critical spectral width introduced in (4.16*a*).

Next we want to comment on the possibility of satisfying the interaction condition

$$\Omega_M^+(\boldsymbol{\kappa}_0) = \Omega_I(\kappa_0). \tag{5.5}$$

Generally $\Omega_M^+(\boldsymbol{\kappa})$, as given by (A 4), is much larger than the frequency $\Omega_I(\kappa)$ of the internal wave. It is seen from (A 6), however, that Ω_M^+ has a zero near $\theta = \frac{1}{2}\pi$ (θ is the angle between $\mathbf{k}$ and $\boldsymbol{\kappa}$). Thus, in the neighbourhood of $\frac{1}{2}\pi$, Ω_M^+ varies (monotonically as a function of θ) from zero to values larger than $\Omega_I(\kappa)$ (which is independent of θ). If therefore some κ_0 is given, and thereby $\Omega_0 = \Omega_I(\kappa_0)$, it is always possible to find a value of θ near $\frac{1}{2}\pi$ such that (5.5) is satisfied. Then to a good approximation l_0 can be expressed as a function of m_0 as

$$\frac{l_0}{k} = 2\left(\frac{(\delta\rho/\rho)\, m_0/k}{1+\coth m_0 d}\right)^{\frac{1}{2}} - \left(\frac{m_0}{k}\right)^2 \left(\frac{1}{4}+\left(\frac{\sigma}{m_0}\right)^2+\frac{\sigma}{m_0}\coth c\right)^{\frac{1}{2}}, \tag{5.6}$$

which has been sketched in figure 3 (for the case $kd\delta\rho/\rho > (\sigma/k)^2+2k^2|A|_0^2$ where the resonance curve crosses the m axis).

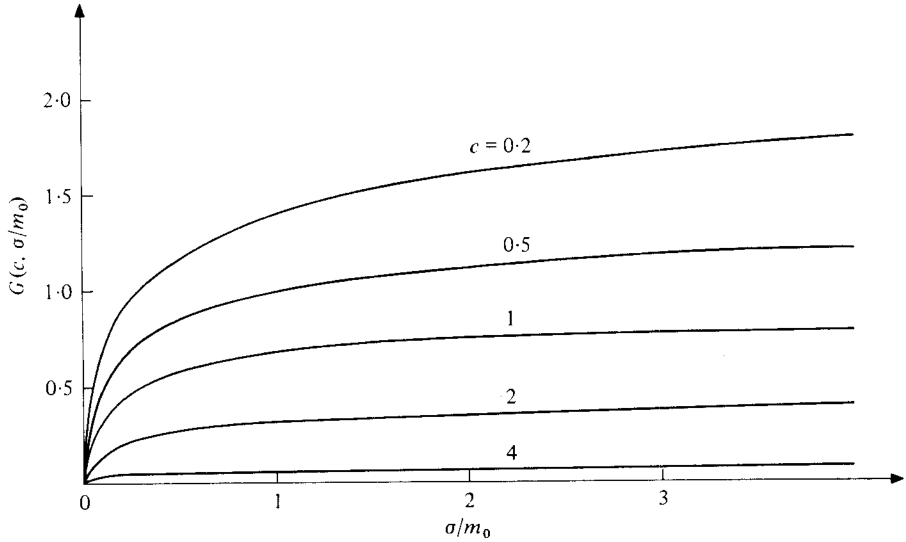

FIGURE 4

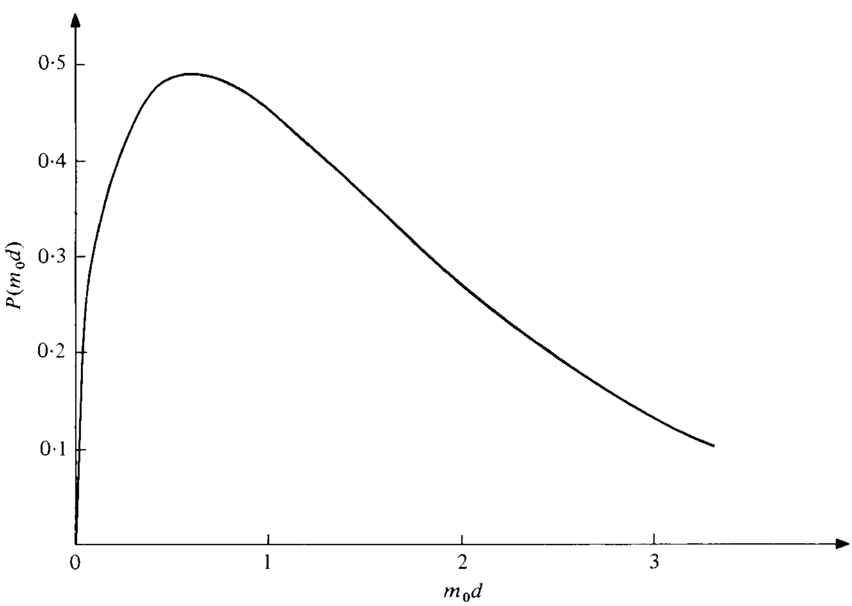

FIGURE 5

Investigating the growth rate, we first note that γ depends rather critically on the spectral energy and angular spread of the surface-wave spectrum through $G(c, \sigma/m_0)$ (see figure 4). To have any appreciable growth, c must be of order unity or less. This introduces the following restriction on the wavenumber m_0 ($c < 6$ say)

$$\frac{m_0}{k} \lesssim 12\frac{k}{\sigma}k^2\overline{|A|_0^2} \equiv 24\frac{k^2\overline{|A|_0^2}}{\Delta\theta}, \tag{5.7}$$

where $\Delta\theta = 2\sigma/k$ is the angular width of the surface-wave spectrum.

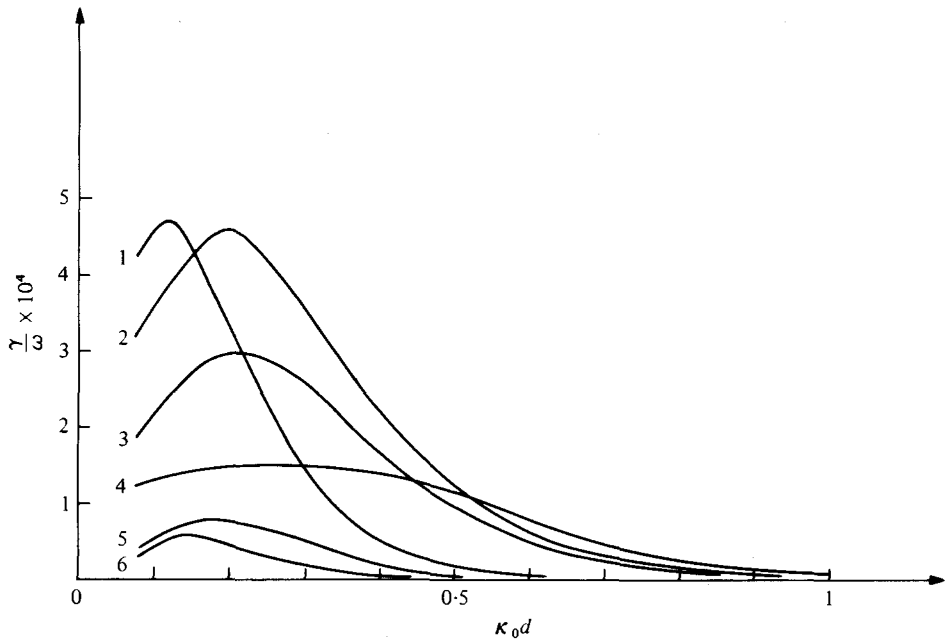

FIGURE 6. Relative growth rates for the modulational interaction as a function of $\kappa_0 z$. The parameter values of $\langle(\nabla\zeta)^2\rangle^{\frac{1}{2}}$, $\Delta\theta$ and kd for the different curves is given in table 1.

Curve no.	1	2	3	4	5	6
$\langle(\nabla\zeta)^2\rangle^{\frac{1}{2}}$	0·066	0·06	0·05	0·05	0·03	0·03
$\Delta\theta$	31°	28°	11°	17°	9°	11°
kd	2	4	2	4	2	4

TABLE 1

On the other hand, the shape of the function $P(m_0 d)$ (see figure 5) tells us that $m_0 d$ should not be too large (< 3, say) and not too small ($> 0{\cdot}05$, say). Combining these requirements, we have roughly the following restriction

$$2 \times 10^{-3} \lesssim kd\, k^2 \overline{|A|_0^2}/\Delta\theta. \tag{5.8}$$

In figure 6 we have calculated the growth rate (or rather γ/ω) as a function of $\kappa_0 d$, for six different sets of values of the parameters $\{k^2\overline{|A|_0^2}\}^{\frac{1}{2}}$, $\Delta\theta$ and kd.

Taking the periods of the surface-wave spectrum to be centred around 8 s, the minimum growth time corresponding to the different curves in figure 6 ranges from approximately 6 hours to 48 hours.

It seems to follow from the numerical examples that rather high values of the r.m.s. wave steepness, and narrow spectral distributions are needed to obtain growth times less than a day.

Finally we note that the description of the last two sections is a linear stability analysis of the 'equilibrium' solution (4.1). Consequently it will not be valid once the spectral perturbation becomes comparable to the initial spectrum.

6. 'Adiabatic' case

In the previous sections we have considered interaction between internal waves and a narrow spectrum of surface waves. Physically that situation probably corresponds to the swell spectrum at a considerable distance from a major storm area.

In this section we address ourselves to the question of interaction between an internal wave and a spectrum of fairly broad bandwidth as found near, or in, a storm area. This problem in its generality is, of course, a formidable one, and its solution is far beyond the modest scope of this paper. If, however, our main interest is the interaction with very long wavelength internal waves, some progress can be made. By long wavelength we mean long compared to wavelength in the surface-wave spectrum. Since the time scales associated with the internal wave (wave period, and growth time) are much larger than the wave periods in the surface-wave spectrum, it is to be expected that some sort of 'adiabatic' approximation can be used.

A fairly general transport equation can be written as follows

$$\frac{\partial J}{\partial t}+\frac{\partial \omega}{\partial \mathbf{k}}\cdot\frac{\partial J}{\partial \mathbf{x}}-\frac{\partial \omega}{\partial \mathbf{x}}\cdot\frac{\partial J}{\partial \mathbf{k}} = S_W+S_B+S_{\mathrm{coll}}, \tag{6.1}$$

where $J(\mathbf{k},\mathbf{x},t)$ denotes the action density equal to the spectral energy density, divided by the intrinsic frequency $(gk)^{\frac{1}{2}}$ (see Willebrand 1974).

On the right-hand side of (6.1) are three source terms which tend to change the wave spectrum. S_w is the input from the wind, S_{coll} represents the statistical four-wave interactions between surface waves, and S_B represents the loss due to breaking and turbulent dissipation.

We shall next assume that some quasi-steady state has been reached in which the right-hand side of (6.1) is unimportant (i.e. the rate of change of J due to the right-hand side is assumed to be on a larger time scale than that of the effect we are going to study). The remaining homogeneous equation

$$\frac{\partial J}{\partial t}+\frac{\partial \omega}{\partial \mathbf{k}}\cdot\frac{\partial J}{\partial \mathbf{x}}-\frac{\partial \omega}{\partial \mathbf{x}}\cdot\frac{\partial J}{\partial \mathbf{k}} = 0, \tag{6.2}$$

where

$$\omega = (gk)^{\frac{1}{2}}+\mathbf{k}\,.\,\nabla_h\bar{\phi}+\iint S(\mathbf{k},\mathbf{k}')\,J(\mathbf{k}')\,d\mathbf{k}',$$

describes how the spectral action density J adjusts adiabatically to slow variations in the average surface flow $\nabla_h\bar{\phi}$.† Note that (6.2) is equivalent to the canonical equations of geometrical optics. The last term in the expression for ω in (6.2) is the averaged nonlinear frequency shift for the spectral component $\mathbf{k}$, due to the presence of the rest of the spectrum (see Longuet-Higgins & Phillips 1962; Willebrand 1974).

The surface flow $\nabla_h\bar{\phi}$, on the other hand, is influenced by non-uniformities of the spectral distribution. A straightforward generalization of (2.7) gives

$$\frac{\partial\bar{\phi}}{\partial z} = \nabla_h\cdot\iint \mathbf{k}\,J(\mathbf{k},\mathbf{x},t)\,d\mathbf{k}, \tag{6.3}$$

where again the term $\partial^2\bar{\phi}/\partial t^2$ has been neglected.

† We have again omitted the averaging sign of $\bar{\phi}$.

While in the narrow-band case of the previous sections the nonlinear frequency shift (implicitly present through the first term on the right-hand side of (2.6)) played an important part, this is no longer expected to be so in the broad-band case. The reason is that, in the narrow-band case with a small variation of group velocities through the spectrum, the presence of a small nonlinear frequency shift makes it possible to have an undamped, or very slightly damped, modulational mode. In the broad-band case, however, there is a large variation of group velocities through the spectrum, so that any broad-band modulations of the spectrum will soon be wiped out by phase mixing, regardless of any small nonlinear frequency shift. For this reason, we are going to neglect the last term in the expression for ω in the following.

The system of equations (2.3), (2.4), (6.2) and (6.3) now describes the interaction between an internal wave and a surface-wave spectrum. A basic solution to the system is again a uniform spectrum $J_0(\mathbf{k})$, and no internal wave $\overline{\phi} = w = 0$.

Investigating the stability of this solution, we perturb as follows:

$$J(\mathbf{k}, \mathbf{x}, t) = J_0(\mathbf{k}) + J_1(\mathbf{k}, \mathbf{x}, t). \tag{6.4}$$

Taking again a Fourier component

$$\exp i(\boldsymbol{\kappa} . \mathbf{x} - \Omega t)$$

of the perturbed quantities J_1, $\overline{\phi}$ and w, one obtains, by the same arguments as in § 4, the same expression for $\overline{\phi}$ and w, where B is now given by

$$B = \frac{1}{\kappa}\frac{\partial\overline{\phi}}{\partial z}(0) = i\frac{\boldsymbol{\kappa}}{\kappa}\cdot \iint \mathbf{k}\, J_1\, d\mathbf{k}. \tag{6.5}$$

Using the same procedure as in § 4, the following dispersion relation is readily deduced

$$H(\Omega, \kappa) = \left\{1 + \coth \kappa d \left(1 - \frac{g\kappa}{\Omega^2}\frac{\partial\rho}{\rho}\right)\right\} \iint \frac{(\boldsymbol{\kappa} . \mathbf{k})^2 (\boldsymbol{\kappa}/\kappa) . \partial J_0/\partial \mathbf{k}}{\Omega - \boldsymbol{\kappa} . \partial\omega/\partial\mathbf{k}}\, d\mathbf{k}, \tag{6.6}$$

where H is given in (4.5). Comparing (6.6) and (4.7) we note that, in contrast to the narrow-band case, there is no longer a modulational mode (for the physical reason explained above). We are left with the internal-wave mode, which is weakly coupled to the surface-wave spectrum.

By assuming that $\Omega = \Omega_I(\kappa) + \delta\Omega$, where $|\delta\Omega| \ll \Omega_I$, we obtain by iteration the following approximate solution of (6.6),

$$\Omega(\boldsymbol{\kappa}) = \Omega_I(\kappa) - \frac{1}{(\partial H/\partial\Omega)\sinh^2 \kappa d}\left[P \iint \frac{(\boldsymbol{\kappa} . \mathbf{k})^2 (\boldsymbol{\kappa}/\kappa) . \partial J_0/\partial k\, d\mathbf{k}}{\Omega_I - \boldsymbol{\kappa} . \partial\omega/\partial\mathbf{k}} + i\pi \iint (\boldsymbol{\kappa} . \mathbf{k})^2 \frac{\boldsymbol{\kappa}}{\kappa} \cdot \frac{\partial J_0}{\partial\mathbf{k}}\, \delta\left(\Omega_I - \frac{\partial\omega}{\partial\mathbf{k}} . \boldsymbol{\kappa}\right) d\mathbf{k}\right]. \tag{6.7}$$

Note that the integral (6.6) is to be interpreted in the same way as for Q in (4.7).

Thus the growth rate, or damping, becomes

$$\gamma = \frac{-\pi}{(\partial H/\partial\Omega)\sinh^2 \kappa d}\iint (\boldsymbol{\kappa} . \mathbf{k})^2 \frac{\boldsymbol{\kappa}}{\kappa} \cdot \frac{\partial J_0}{\partial\mathbf{k}}\, \delta\left(\Omega_I - \frac{\partial\omega}{\partial\mathbf{k}} . \boldsymbol{\kappa}\right) d\mathbf{k}. \tag{6.8}$$

The contribution to the integral thus comes from the curve in the $\mathbf{k}$ plane where the resonance condition $\Omega_I(\kappa) = \boldsymbol{\kappa} . \partial\omega/\partial\mathbf{k}$ is satisfied. Bearing in mind that v_g is much larger than Ω_I/κ (except perhaps for the small-wavelength part of the spectrum), we can write the resonance condition approximately

$$\frac{\Omega_I(\kappa)}{\kappa} = \frac{g^{\frac{1}{2}}}{2}\frac{k_\parallel}{k_\perp^{\frac{3}{2}}} \tag{6.9}$$

where $k_\parallel$ and $k_\perp$ are the components parallel and perpendicular to $\boldsymbol{\kappa}$, respectively. Using (6.9) and the expression for $\partial H/\partial \Omega$, (6.8) can be simplified to

$$\frac{\gamma}{\Omega_I} = 4\pi\frac{\delta\rho}{\rho}e^{-2\kappa d}\int\left(\frac{k_\perp^9}{g}\right)^{\frac{1}{2}}\frac{\partial J_0}{\partial k_\parallel}dk_\perp, \tag{6.10}$$

where $k_\parallel$ is given as a function of $k_\perp$ by (6.9).

Considering fairly general spectra $J_0(\mathbf{k})$, it can be shown from (6.10) that one can always get $\gamma > 0$ when the direction of the internal wave is chosen within some finite angular domain with respect to the main direction of the spectrum ('wind direction').

The growth rate, however, is discouragingly small for realistic spectral energy densities.

To see the connection between this rather negative result and the findings of Olbers & Herterich (1979) cited in the introduction, we start from a spectral transfer equation, similar to their equation (3.5) (see also Davidson 1972, cha. 13) namely

$$\begin{aligned}\frac{\partial}{\partial t}J^I(\boldsymbol{\kappa}) = 4\pi\int |V^{ISS}_{(\boldsymbol{\kappa},\mathbf{k}_1,\mathbf{k}_2)}|^2\{J^I(\boldsymbol{\kappa})\,(J(\mathbf{k}_2)-J(\mathbf{k}_1))-J(\mathbf{k}_1)\,J(\mathbf{k}_2)\}\\ \times\,\delta(\boldsymbol{\kappa}-\mathbf{k}_1+\mathbf{k}_2)\,\delta(\Omega^I(\kappa)-\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2))\,d\mathbf{k}_1\,d\mathbf{k}_2,\end{aligned} \tag{6.11}$$

where J^I and J are the action densities of the internal- and surface-wave mode, respectively, and $V^{ISS}_{(\boldsymbol{\kappa},\mathbf{k}_1,\mathbf{k}_2)}$ is the coupling coefficient of the three-wave interaction between an internal wave $\boldsymbol{\kappa}$, $\Omega^I(\kappa)$ and two surface waves $\mathbf{k}_1, \omega(\mathbf{k}_1)$ and $\mathbf{k}_2, \omega(\mathbf{k}_2)$.†

In the adiabatic limit of (6.11), when $\kappa \ll |\mathbf{k}_1|, |\mathbf{k}_2|$, we have approximately

$$\frac{\partial}{\partial t}J^I(\boldsymbol{\kappa}) = \gamma(\boldsymbol{\kappa})\,J^I(\boldsymbol{\kappa}) - 4\pi\int |V^{ISS}_{(\boldsymbol{\kappa},\mathbf{k}'+\boldsymbol{\kappa},\mathbf{k}')}|^2 J(\mathbf{k}')\,J(\mathbf{k}'+\boldsymbol{\kappa})\,\delta\left((\Omega^I(\kappa)-\boldsymbol{\kappa}\,.\,\frac{\partial\omega}{\partial\mathbf{k}'}\right)d\mathbf{k}', \tag{6.12}$$

where $$\gamma(\boldsymbol{\kappa}) = -4\pi\int |V^{ISS}_{(\boldsymbol{\kappa},\mathbf{k}'+\boldsymbol{\kappa},\mathbf{k}')}|^2\boldsymbol{\kappa}\,.\,\frac{\partial J(\mathbf{k}')}{\partial\mathbf{k}'}\,\delta\left(\Omega^I(\kappa)-\boldsymbol{\kappa}\,.\,\frac{\partial\omega}{\partial\mathbf{k}'}\right)d\mathbf{k}'.$$

If the second term of the right-hand side of (6.12) is neglected one obtains the same result as (6.8) when the coupling coefficient is identified as

$$|V^{ISS}_{(\boldsymbol{\kappa},\mathbf{k}+\boldsymbol{\kappa},\mathbf{k})}|^2 = \frac{(\boldsymbol{\kappa}\,.\,\mathbf{k})^2}{4\kappa\sinh^2\kappa d\,\partial H/\partial\Omega}. \tag{6.13}$$

When, however, $J \gg J^I$, it is seen from (6.11) that the last term of the bracket { } dominates. This is exactly the term that Olbers & Herterich use to calculate the energy transfer to the internal-wave spectrum. In the adiabatic limit (neglecting the first term on the right-hand side of (6.12) and using (6.13)) we have

$$\frac{\partial}{\partial t}J^I(\kappa) = -\pi\int\frac{(\boldsymbol{\kappa}\,.\,\mathbf{k})^2\,\delta(\Omega^I(\kappa)-\boldsymbol{\kappa}\,.\,\partial\omega/\partial\mathbf{k})}{\kappa\sinh^2\kappa d\,\partial H/\partial\Omega}J(\mathbf{k})\,J(\mathbf{k}+\boldsymbol{\kappa})\,d\mathbf{k}. \tag{6.14}$$

† Note that the coupling coefficients refer to a situation where the complex amplitudes A of the waves are chosen such that $|A|^2$ is the action density.

We conclude that for the broad-band case studied in this section the 'modulational' mechanism (*c*) is contained in the incoherent three-wave interaction mechanism (*b*). The main contribution to the energy transfer to the internal-wave spectrum (for $J \gg J^I$) comes from (*b*) and not from (*c*).

This does not mean that the equations (6.2) and (6.3), describing the modulational effect in the broad-band case, are useless. When consideration is given to the more general equation (6.11), the system (6.2) and (6.3) can be used to simplify the calculations in the adiabatic limit as shown by (6.14).† Another example is provided by the calculation of the distortion of a surface-wave spectrum by a well-developed spectrum of internal waves. In the adiabatic limit the equation describing the slow distortion of $J(\mathbf{k}, t)$, is simply a diffusion equation (in $\mathbf{k}$ space). The details of derivation are very similar to the derivation of the so-called 'quasi-linear' plasma theory (see Davidson 1972; and also Vedenov, Gordeev & Rudakov 1967) and will not be presented here. The result is the equation

$$\frac{\partial J}{\partial t} = \frac{\partial}{\partial \mathbf{k}} \cdot \left(\mathbf{D} \cdot \frac{\partial J}{\partial \mathbf{k}}\right), \tag{6.15}$$

where the 'diffusion' tensor $\mathbf{D}$ is given in terms of the spectrum of surface induced current $\langle \mathbf{u}(\boldsymbol{\kappa})\, \mathbf{u}^*(\boldsymbol{\kappa}) \rangle$ (due to the internal waves) as

$$\mathbf{D} = \pi \int (\mathbf{k} \cdot \boldsymbol{\kappa})^2 \langle \mathbf{u}(\boldsymbol{\kappa})\, \mathbf{u}^*(\boldsymbol{\kappa}) \rangle \, d\boldsymbol{\kappa}.$$

7. Discussion and conclusions

We have presented theoretical models for a coupling of an internal wave of the lowest mode with a surface-wave spectrum.

As shown schematically in figure 1, the interaction can be interpreted physically in the following way:

(i) The non-uniform surface current induced by an internal wave causes refraction, and thereby modulation of the surface-wave spectrum.

(ii) This modulation introduces a non-uniform radiation stress, and thereby forces doing work on the internal wave.

In §§ 2–5 we studied this interaction for a narrow-band spectrum, and found that it could be considered as a mode coupling between a modulational mode, and an internal wave mode. An instability may occur when the angular distribution of the surface-wave spectrum is such as to support modulational modes nearly perpendicular to the main surface-wave direction. In that case a growing modulation and internal wave propagate nearly perpendicular to the wave direction, with the same frequency and wavenumber. In order to present our analysis of the interaction as clearly as possible, we have chosen a rather simple model for the angular distribution of the surface-wave spectrum. This permits us to calculate the growth rate explicitly. In order to do the calculations for a wider class of angular distributions, a numerical approach will probably be necessary due to the complexity of the dispersion relation (A 1).

† The usefulness of this procedure does of course depend on the validity of neglecting the nonlinear frequency shift, as explained above.

The numerical examples seems to indicate that rather large r.m.s. wave steepness, and a narrow angular distribution are needed for growth times less than a day (for the conditions of the seasonal thermocline, $\delta\rho/\rho \sim 10^{-3}$, $d \sim$ 50–100 m). For example, taking $\langle(\nabla\zeta)^2\rangle^{\frac{1}{2}} = 0{\cdot}06$, $\Delta\theta = 30°$, and $kd = 4$, the growth time is roughly 6–8 hours, while, for the values 0·03, 10° and 4, the growth time is roughly 2 days.

This probably indicates that the conditions for an appreciable growth rate is too limiting for the instability to be of geophysical significance.

For the broad-band case discussed in § 6, we find that the growth rate of the modulational instability is very small indeed. We established the connection with the incoherent three-wave interaction (*b*), and showed that the modulational mechanism (in the broad-band case) is contained in (*b*).

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Appendix

In this section we solve the dispersion relation (4.11) for the simple spectrum (4.17). For the stable case ($\mathrm{Im}\,\Omega < 0$) $Q(\Omega, \boldsymbol{\kappa})$ is defined as†

$$Q(\Omega,\boldsymbol{\kappa}) = \omega k^2 \int_{-\infty}^{\infty} \frac{\tilde{F}_0(p_\xi+\kappa_\xi/2)-\tilde{F}_0(p_\xi-\kappa_\xi/2)}{\Omega - v_g l + \beta p_\xi}\, dp_\xi + \frac{2\pi i}{\beta}\left[\tilde{F}_0\left(-\frac{\Omega-v_g l}{\beta}+\tfrac{1}{2}\kappa_\xi\right) - \tilde{F}_0\left(-\frac{\Omega-v_g l}{\beta}-\tfrac{1}{2}\kappa_\xi\right)\right]. \quad \text{(A 1)}$$

When $\mathrm{Im}\,\Omega = 0$, the integral is to be interpreted as the Cauchy principal value, and the factor 2 of the square bracket should be deleted.

For the 'water bag' model (4.17) the integral in (A 1) can easily be evaluated, and the square bracket vanishes provided

$$\left|\frac{\Omega - v_g l}{\beta} \pm \frac{\kappa_\xi}{2}\right| > \sigma. \quad \text{(A 2)}$$

The dispersion relation (4.11) then becomes

$$2c \equiv \frac{2\beta\sigma}{\omega k^2 \overline{|A|_0^2}} = \ln\left|\frac{((\Omega - v_g l)/\beta)^2 - (\tfrac{1}{2}\kappa + \sigma)^2}{((\Omega - v_g l)/\beta)^2 - (\tfrac{1}{2}\kappa_\xi - \sigma)^2}\right|. \quad \text{(A 3)}$$

which is solved to produce

$$\Omega_M^{\pm} = v_g l \pm \beta(\tfrac{1}{4}\kappa_\xi^2 + \sigma^2 - \sigma\kappa_\xi \coth c)^{\frac{1}{2}}. \quad \text{(A 4)}$$

It is a simple exercise to show that in the limit $\sigma \to 0$ (A 4) tends to the result (4.12) when the identification $2\overline{|A|_0^2} = a_0^2$ is made.

Using (A 4), the condition (A 2) can be put in the form

$$|(x^2 + 1 - 2x \coth c)^{\frac{1}{2}} \pm x| > 1, \quad \text{(A 5)}$$

where $x = \kappa_\xi/2\sigma$. With $\coth c > 1$ (A 5) is seen to be satisfied for $x < 0$, i.e. for $\kappa_\xi < 0$. This is exactly the domain of prime interest to us since $\kappa_\xi < 0$ when $\theta > 35{\cdot}26°$.

† Alber (1978) seems to have missed out the factor β^{-1} of the square bracket in (A 1).

When θ is near $\frac{1}{2}\pi$, (A 4) can be written

$$\Omega_M^{\pm} = v_g \left\{ l \pm \frac{m^2}{k} \left(\tfrac{1}{4} + \left(\frac{\sigma}{m}\right)^2 + \frac{\sigma}{m} \coth c \right)^{\frac{1}{2}} \right\}. \tag{A 6}$$

Lastly we note that, when θ is near $\frac{1}{2}\pi$, c can be expressed as follows

$$c = 2\sigma m / \sigma_0^2, \tag{A 7}$$

where σ_0 is the critical spectral width in the wave direction as defined in (4.16*a*).

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